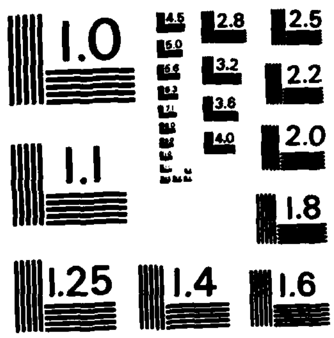

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ON THE MAXIMUM-WEIGHT CLIQUE PROBLEM

by

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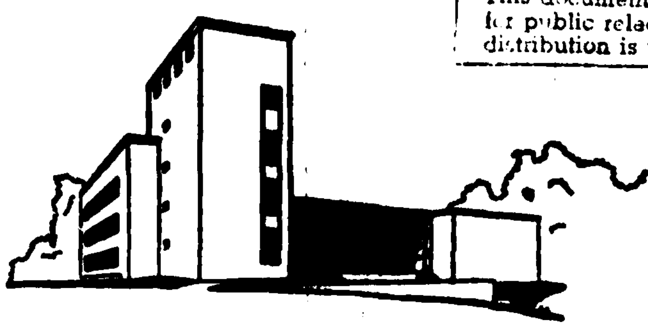

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Management Science Research Report No. MSRR 518

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June 1985

The research of the first author was supported by Grant ECS-8205425 of the National Science Foundation and Contract N00014-82-K-0329 NR047-607 with the U.S. Office of Naval Research. Reproduction in whole or in part is permitted for any purpose of the U.S. Government.

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ABSTRACT

~~We~~ The authors introduce several new classes of graphs on which the maximum-weight clique problem is solvable in polynomial time. Their common feature, and the central idea of ~~our~~ the algorithms, is that every clique of any of ~~our~~ these graphs is contained in some member of a polynomial-sized collection of induced subgraphs that are complements of bipartite graphs. ~~Our~~ This approach is quite general, and might conceivably yield many other classes of graphs along with corresponding polynomial time algorithms. Additional keywords: Triangulated graphs; triangle free graphs; Theorems.

0. INTRODUCTION

The maximum-weight clique problem, or MWCP for short, goes as follows: given a graph whose vertices carry numerical weights, find a clique (that is, a set of pairwise adjacent vertices) whose total weight is as large as possible. This problem is notoriously hard, even when all the weights are equal; the problem of deciding whether a prescribed graph contains a clique of a prescribed size is NP-complete; in fact, this problem was one of the five prototypes of NP-complete problems presented by Cook (1971) in the classical paper that laid the foundations of NP-completeness theory.

Nevertheless, there are several known polynomial-time algorithms each of which solves the MWCP on all graphs coming from some restricted class. In this paper, we present new algorithms of this kind. The method used to design these algorithms is quite general, and might conceivably yield many other algorithms.

1. THE RESULTS

We reserve the letter n for the number of vertices of a graph G. If C is a class of graphs and t is a number, we say that C is t-bounded if every G in C has $O(n^t)$ maximal cliques. (Here, as usual, "maximal" is meant with respect to set inclusion, not size.) We say that C is tame if no G in C contains an induced subgraph that is an odd antihole (defined as the complement of an odd hole, i.e. of a chordless cycle whose length is odd and at least five). We say that a green/red coloring of the edges of a graph is C-formative if the green graph belongs to C and the red graph contains no triangle. Our key result goes as follows.

THEOREM 1. For every t-bounded and tame class C there is an algorithm that, given any graph G along with a C-formative coloring of its edges, solves any MWCP on G in $O(n^{2t+3})$ steps.||

Our prime example of a t-bounded and tame class of graphs is the class TR of <u>triangulated</u> graphs, defined as graphs in which every cycle of length at least four has a chord. These graphs were introduced by Hajnal and Surányi (1958) and studied further by Berge (1960), Dirac (1961), and many others. In particular, Dirac proved that every triangulated graph has a <u>simplicial</u> vertex, defined as a vertex whose neighbours are all adjacent to each other; an instant corollary of this theorem states that every triangulated graph has at most n maximal cliques. Hence TR is 1-bounded; since every odd antihole contains a chordless cycle of length four or five, TR is also tame.

Now let C^1 stand for the class of all graphs that admit C-formative edge colorings. Theorem 1 provides a polynomial-time algorithm for solving any MWCP on any graph in some subclass C* of C^1 if and only if a polynomial-time algorithm to find a C-formative edge coloring of every graph in C* is available. In particular, if such an algorithm is available for every graph in C^1, that algorithm can be used to test membership in C^1. Our next result shows that the existence of such an algorithm is unlikely when C = TR.

THEOREM 2. Testing membership in TR^1 is an NP-complete problem.||

Thus we are led to look for some proper subclass C* of C^1 with a polynomial-time algorithm to construct a C-formative edge coloring of every graph in C*. One way of doing this is to impose additional constraints on the C-formative edge colorings. For instance, rather than requiring that the red graph contain no triangle, we might insist that it be bipartite. This idea leads to the following notion: we say that a partition of the vertex set of G into disjoint parts V_1 and V_2 is <u>C-formative</u> if each of the two subgraphs of G

induced by V_1 and V_2 belongs to C. Clearly, as long as C is closed under disjoint unions, each C-formative vertex partition of G yields a C-formative edge coloring of G. In this case, Theorem 1 has an instant collary with "coloring of its edges" replaced by "partition of its vertices". In fact, the assumption that C is tame can be dropped, and the resulting statement holds true even if C is not closed under disjoint unions.

THEOREM 3. For every t-bounded class C there is an algorithm that, given any graph G along with a C-formative partition of its vertices solves any MWCP on G in $O(n^{2t+3})$ steps.||

Let C^2 stand for the class of all graphs that admit C-formative vertex partitions. We have observed that $C^2 \subseteq C^1$ whenever C is closed under disjoint unions; in particular, $TR^2 \subseteq TR^1$. Examples of graphs in TR^2-TR are all holes and antiholes. For holes this is obvious. For an antihole, let V_1 and V_2 be the odd-numbered and even-numbered vertices, respectively, of the cycle whose complement is the given antihole. Then V_1 and V_2 is a TR-formative vertex partition of the antihole. Examples of graphs in TR^1-TR^2 are all graphs that contain no triangles and whose chromatic number exceeds four. (Triangle-free graphs of an arbitrarily high chromatic number were constructed first by Tutte, writing under the name of Blanche Descartes (1954); another family of such graphs was constructed independently by Mycielski (1955). For a strong result on such graphs, see Erdős (1959) and Lovász (1968).) Trivially, every such graph G belongs to TR^1 (color all edges red). To see that $G \notin TR^2$, observe that every triangulated induced subgraph of G is a forest; since the chromatic number of G exceeds four, two such subgraphs cannot cover all the vertices of G.

Again, Theorem 3 provides a polynomial-time algorithm for solving any MWCP on any graph in some subclass C* of C^2 if any only if a polynomial-time algorithm to find a C-formative vertex partition of every graph in C* is available. Again, if such an algorithm is available for every graph in C^2, then the algorithm can be used to test membership in C^2. Again, we have a result showing that the existence of such an algorithm is unlikely when C = TR.

THEOREM 4. Testing membership in TR^2 is an NP-complete problem.||

To obtain a subclass C* of C^2 such that a C-formative vertex partition of every graph in C* can be found in polynomial time, we may begin with any polynomial-time heuristic that <u>attempts</u> to construct the partition, and then simply define C* as the class of those graphs on which the heuristic <u>succeeds</u>. (The same approach can of course be used to obtain a subclass of C^1 such that a C-formative edge coloring of every graph in the subclass can be found in polynomial time.) We use this approach with heuristics based on a certain subroutine that we call GREEDY. The input of GREEDY is any graph G whose vertices have been labeled as $v_1, v_2, \ldots, v_n$; its output is either a C-formative vertex partition of G or a failure message. In the description of GREEDY and later on, we let G(S) denote the subgraph of G induced by S.

```
GREEDY:
V1 ← Φ, V2 ← Φ
for i = 1,2,...,n do
    if  G(V1 ∪ {vi}) ε C then V1 ← V1 ∪ {vi}
                         else V2 ← V2 ∪ {vi}
    endif
endfor
    if  G(V2) ε C then return V1 and V2
                  else return a failure message endif
```

Note that GREEDY runs in time $O(n^{s+1})$ whenever membership in C can be tested in $O(n^s)$ steps for some constant s.

Since GREEDY works with labeled graphs, it gives rise to a variety of algorithms working with unlabeled graphs: each of these algorithms first labels the vertices and then applies GREEDY. In the remainder of this section, we shall discuss three special cases in detail.

First, the simplest way to construct a labeling of the vertices of G is to take an arbitrary labeling; we shall let C^3 denote the class of graphs on which the resulting algorithm always delivers a C-formative vertex partition. To put it differently, G belongs to C^3 if and only if GREEDY succeeds on G for each of its n! labelings.

THEOREM 5. Let C be any t-bounded class of graphs such that members of C can be recognized in $O(n^{2t+2})$ steps. Then any MWCP on any graph in C^3 can be solved in $O(n^{2t+3})$ steps.||

Note that the hypothesis of Theorem 5 is satisfied when C=TR and t=1: Dirac's theorem implies at once that triangulated graphs can be recognized in $O(n^4)$ steps. (Actually, the runnning time of an algorithm designed by Rose, Tarjan, and Lueker (1976) to recognize triangulated graphs is only $O(n^2)$, but $O(n^4)$ is good enough for our purpose.)

COROLLARY 5A. Any MWCP on any graph in TR^3 can be solved in $O(n^5)$ steps.||

Our second algorithm attempts to construct a labeling whose properties guarantee the success of GREEDY. If such a labeling is found then GREEDY is applied; else a failure message is returned. To explain the details, we need two more definitons. First, a vertex v in a graph F will be called <u>C-acceptable</u> if F-v contains no disjoint sets S_1, S_2 of vertices such that

$$F(S_1) \in C,\ F(S_2) \in C,\ F(S_1 \cup \{v\}) \notin C,\ F(S_2 \cup \{v\}) \notin C.$$

Second, a labeling $v_1, v_2, \ldots, v_n$ of the vertices of G will be called C-formative if each v_k is C-acceptable in $G(\{v_1, v_2, \ldots, v_k\})$. Clearly, if GREEDY is given a graph G with a C-formative labeling, then it finds a C-formative vertex partition of G. (Here, we are tacitly assuming that graphs with no vertices at all belong to C.) The following algorithm, given any graph G, will either find a C-formative labeling of G or establish that no such labeling exists.

```
LABEL:
F←G, k←n, failure←false
While k > 0 and failure = false do
   if F has a C-acceptable vertex v
      then  v_k←v, F←F-v, k←k-1
      else  failure←true
   endif
endwhile
if k = 0 then return v_1, v_2, ..., v_n
        else return a failure message
endif
```

Note that LABEL runs in time $O(n^{s+2})$ whenever C-acceptable vertices can be recognized in $O(n^s)$ steps for some constant s.

Let C^4 denote the class of all graphs on which LABEL succeeds. (This class is well-defined: the success of LABEL is independent of the choice of v in each iteration.)

THEOREM 6. Let C be any t-bounded class of graphs such that memebers of C can be recognized in $O(n^{2t+2})$ steps, and C-acceptable vertices can be

recognized in $O(n^{2t+1})$ steps. Then any MWCP on any graph in C^4 can be solved in $O(n^{2t+3})$ steps.||

Trivially, a vertex v in a graph F is TR-acceptable if and only if F-v contains no disjoint sets S_1, S_2 of vertices such that both $F(S_1 \cup \{v\})$ and $F(S_2 \cup \{v\})$ are chordless cycles of length at least four. To put it differently, v is <u>not</u> TR-acceptable if and only if F-v contains vertex disjoint paths P_1, P_2 such that each P_i has at least two edges, and its terminal points x_i, y_i are adjacent to v in F. For each fixed choice of x_1, y_1, x_2, y_2, the existence of P_1, P_2 can be tested by efficient algorithms designed independently by Seymour (1980) and Shiloach (1980). In particular, Shiloach's algorithm runs in time $O(n^3)$; it follows that TR-acceptable vertices can be recognized in $O(n^7)$ steps.

COROLLARY 6A. Any MWCP on any graph in TR^4 can be solved in $O(n^9)$ steps.||

Note that $TR^3 \not\subseteq TR^4$ and $TR^4 \not\subseteq TR^3$: the antihole with seven vertices belongs to TR^3-TR^4 and the graph shown in Fig. 1 belongs to TR^4-TR^3.

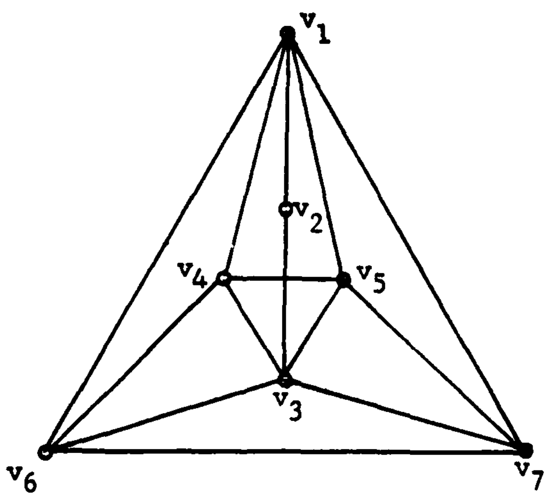

Figure 1

However, Corollary 6A holds with TR^4 replaced by a certain class TR^5 such that

$$TR^3 \cup TR^4 \subset TR^5 \subset TR^2.$$

More generally, consider an arbitrary class C of graphs. LABEL, given an arbitrary graph G, will produce some induced subgraph F of G and a labeling $v_{k+1}, v_{k+2}, \ldots, v_n$ of the vertices in G-F. It is an easy exercise to show that F depends only on G and C; we set $G \in C^5$ if and only if $F \in C^3$. Trivially, a C-formative vertex partition of any G in C can be constructed by first taking an arbitrary labeling $v_1, v_2, \ldots, v_k$ of the vertices of F and then applying GREEDY.

THEOREM 7. Let C be any t-bounded class of graphs such that members of C can be recognized in $O(n^{2t+2})$ steps, and C-acceptable vertices can be recognized in $O(n^{2t+1})$ steps. Then any MWCP on any graph in C^5 can be solved in $O(n^{2t+3})$ steps.||

COROLLARY 7A. Any MWCP on any graph in TR^5 can be solved in $O(n^9)$ steps.||

Examples of graphs in $TR^5 - (TR^3 \cup TR^4)$ can be obtained by joining a graph in TR^4 to a graph in TR^3 by an appropriate set of edges. For instance, joining the vertices 1, 6 and 7 of the graph in Figure 1 by three edges to any three vertices of the antihole on seven vertices yields a graph in TR^5.

As we have observed, $C^5 \subseteq C^2$; since $G \in C^4$ if any only if F has no vertices at all, we have $C^4 \subseteq C^5$. Finally, if C is <u>hereditary</u> in the sense that every induced subgraph of every graph in C is also in C, then C^3 is hereditary, and so $C^3 \subseteq C^5$.

Variations on the theme of C^5 abound. For instance, observe that all bipartite graphs and complements of all bipartite graphs belong to TR^2; in fact, a TR-formative vertex partition of each of these graphs can be found in

$O(n^2)$ steps. This observation suggests setting $G \in TR^6$ if and only if $F \in TR^3$ or F is bipartite or the complement of F is bipartite.

THEOREM 8. Any MWCP on any graph in TR^6 can be solved in $O(n^9)$ steps.||

Trivially, we have $TR^5 \subseteq TR^6 \subseteq TR^2$. Examples of graphs in TR^6-TR^5 can be obtained by joining a graph in TR^4 to a bipartite graph not in $TR^3 \cup TR^4$ by an appropriate set of edges. For instance, joining the vertices 1, 6, 7 of the graph shown in Fig. 1 by three edges to any three vertices of $K_{5,5} - e$ (where e is an arbitrary edge of the complete bipartite graph $K_{5,5}$) yields a graph in TR^6-TR^5. As examples of graphs in TR^2-TR^6, we mention the antiholes with n vertices for $n \geq 9$.

In closing this section, we recall that members of TR^4 are recognizable in $O(n^9)$ steps. We know no polynomial-time algorithm for testing membership in TR^3 or TR^5 or TR^6, although for each of these classes there is a polynomial time algorithm that, given any graph G, solves the MWCP on G or shows that G does not belong to the class.

2. PROOFS

We begin with two well-known facts; their proofs are included for the sake of completeness.

FACT 1. There is an algorithm that, given any graph G, lists all the maximal cliques in G in $O(n^2k^2)$ steps, with k standing for the number of items on the list.

PROOF. The following algorithm will do.

```
V←the vertex set of G; LIST←{∅}.
while  V ≠ ∅  do
    remove a vertex w from V
    N←the set of all neighbours of w that lie outside V
    t←0
    for all Q in LIST do
        P←Q ∩ N, t←t+1, Q_t←P ∪ {w}
        if P = Q then  LIST←LIST - {Q}  endif
    endfor
    for j = 1,2,...,t, do
        SMALL(j)←false
            for i = 1,2,...,j-1 do
                if  Q_i ⊆ Q_j  then  SMALL(i)←true  endif
                if  Q_j ⊂ Q_i  then  SMALL(j)←true  endif
            endfor
    endfor
    for j = 1,2,...,t do
        if SMALL(j) = false then  LIST←LIST ∪ {Q_j}  endif
    endfor
endwhile.||
```

Actually, the running time of an algorithm designed by Tsukiyama, Ide, Arioshi and Shirokawa (1977) to list all the maximal cliques in G is only $O(n^3k)$, but $O(n^2k^2)$ is good enough for our purpose.

FACT 2. There is an algorithm that, given any graph G along with two cliques in G whose union contains all the vertices of G, solves any MWCP on G in $O(n^3)$ steps.

PROOF. The MWCP reduces to the problem of finding a minimum capacity cut in a network N constructed as follows. No generality is lost by assuming that the two cliques, Q_1 and Q_2, that cover all the vertices of G are disjoint. For nodes of N, take all nodes of G and two extra nodes, s and t; for arcs, take

(*) an arc si of capacity equal to the weight of i for each i in Q_1,

(*) an arc jt of capacity equal to the weight of j for each j in Q_2,

(*) an arc ij of infinite capacity for each choice of i in Q_1 and j in Q_2 such that i and j are not adjacent in G.

A _cut_ in a network with source s and sink t is any set S of nodes such that $s \in S$ and $t \notin S$; the _capacity_ of this cut is the sum of the capacities of all the arcs ij with $i \in S$, $j \notin S$. In our network, the capacity of a cut S is finite if and only if the set Q defined by

$$Q = (Q_1 \cap S) \cup (Q_2 \setminus S)$$

is a clique of G; if this is the case then the weight of Q and the capacity of S add up to the total weight of all the vertices of G. Thus finding a clique of maximum weight in G amounts to finding a cut of minimum capacity in N. The latter problem can be solved by a variety of efficient algorithms; in particular, an algorithm designed by Malhotra, Kumar, and Maheshwari (1978) runs in time $O(n^3)$; for details of its implementation, see Chvátal (1983, pp. 380-386).||

Our key notion is that of a _clique basis_, defined as a set of cliques $Q_1, Q_2, \ldots, Q_k$ in a graph G such that each clique in G is a subset of some $Q_i \cup Q_j$. Our key observation goes as follows.

FACT 3. There is an algorithm that, given any clique basis $Q_1, Q_2, \ldots, Q_k$ in any graph G, solves any MWCP on G in $O(n^3k^2)$ steps.

(A proof is hardly required: to solve the MWCP on G, we only need to solve $k(k-1)/2$ problems on the subgraphs of G induced by $Q_i \cup Q_j$.)

Our theorems will be proved in a permuted order.

PROOF OF THEOREM 1. By virtue of Facts 1 and 3, we only need to show that all the maximal cliques in the green graph consitute a clique basis in G. For this purpose, consider an arbitrary clique Q in G and let F be the subgraph of the red graph induced by Q. By assumption, F contains no triangle; since Q is a clique in G and since C is tame, F contains no odd hole. Thus F is bipartite; to put it differently, Q is covered by two cliques of the green graph.||

PROOF OF THEOREM 3. We only need to observe that all the maximal cliques in the subgraphs of G induced by V_1 and V_2 constitute a clique basis in G.||

PROOF OF THEOREM 4. Lovász (1973) proved that the problem of recognizing bicolorable hypergraphs is NP-complete, and that it remains NP-complete even when the input is restricted to hypergraphs with all edges of size three. (A hypergraph is a collection of sets $E_1, E_2, \ldots, E_m$ called edges whose elements are called points; the hypergraph is called bicolorable if its points can be colored black and white in such a way that no edge is monochromatic.) It follows easily that recognizing bicolorable hypergraphs is an NP-complete problem even when the input is restricted to hypergraphs with all edges of size four. Given any such hypergraph H, we shall construct a graph G such that $G \in TR^2$ if and only if H is bicolorable.

First, let us construct a graph F with distinguished vertices x and y such that $F \in TR^2$ and such that x and y must belong to distinct parts of every TR-formative partition of the vertex set of F. This is easy to do: take a $K_{3,3}$ (the complete bipartite graph with three vertices in each part),

add two nonadjacent vertices x and y, and join each of these two by edges to all the vertices of the $K_{3,3}$.

To construct G, we create a copy of F for every pair (E,p), where E is an edge of H and $p \in E$; the two distinguished vertices of this F will be labeled x(E,p) and y(E,p). Then for each p, we identify all the vertices labeled y(E,p), y(E',p), y(E",p),... and label the resulting single vertex p*. Finally, for each edge E of H, we enumerate the elements of E as a,b,c,d and create a chordless cycle of length four in G by joining each of x(E,a), x(E,b) to each of x(E,c), x(E,d).

To see that $G \in TR^2$ only if H is bicolorable, consider any TR-formative partition of the vertices of G and, referring to the two parts as "black" and "white", assign to each p in H the color of p* in G. We only need to show that no edge E of H is monochromatic. For this purpose, enumerate the elements of E as in the construction of G and note that the vertices x(E,a), x(E,b), x(E,c), x(E,d), inducing a chordless cycle of length four, cannot all have the same color. Since the color of each x(E,p) differs from that of p*, the desired conclusion follows.

To see that H is bicolorable only if $G \in TR^2$, consider any bicoloring of H. Transfer the color of each p in H to p* in G, give each x(E,p) the color that differs from the color of p* and, in each F used in the construction of G, color one part of the $K_{3,3}$ black and the other part white. Clearly, each of the two color classes induces a forest in G.||

PROOF OF THEOREM 2. Given any graph H we shall construct a graph G such that $G \in TR^1$ if and only if $H \in TR^2$.

First, let us construct a graph F with a distinguished edge xy such that $F \in TR^1$ and xy must be green in every TR-formative edge coloring of F. This can be done by taking any graph F_0 of a chromatic number greater than six that

contains no triangle, adding two adjacent vertices x and y, and joining each of these two vertices by edges to all the vertices of the F_0. (Trivially, $F \in TR^1$: color all the edges of F_0 red, and all the remaining edges of F green. To see that xy must be green in every TR-formative edge coloring of F, assume the contrary. Writing $z \in V_x$ if zx is green, and $z \in V_y$ if zy is green, observe that each vertex of F_0 belongs to $V_x \cup V_y$, and that each of the three graphs induced in F_0 by $V_x \cap V_y$, $V_x - V_y$ and $V_y - V_x$ is bipartite. This contradicts the fact that the chromatic number of F_0 exceeds six.)

To construct G, we take disjoint graphs F_1, F_2, F_3, H such that each F_i is a copy of F with distinguished edge $x_i y_i$. Then we identify y_1 with x_2, identify y_2 with x_3, add edge $x_1 y_3$, and join each of the two vertices x_1, y_3 by edges to all the vertices of H.

To see that $G \in TR^1$ only if $H \in TR^2$, consider any TR-formative edge coloring of G. Since $x_1 y_1$, $x_2 y_2$ and $x_3 y_3$ are all green, $x_1 y_3$ must be red. For each vertex v of H, write $v \in V_x$ if vx_1 is green, and write $v \in V_y$ if vy_3 is green. Clearly, each vertex of H belongs to precisely one of these two sets (else vx_1y_3 would be a red triangle or $vx_1x_2x_3y_3$ would be a green chordless cycle) and each edge of H with both endpoints in the same set is green. It follows that V_x and V_y form a TR-formative vertex partition of H.

A straightforward reversal of this argument shows that every TR-formative vertex partion of H yields a TR-formative edge coloring of G; hence $H \in TR^2$ only if $G \in TR^1$.||

Theorems 5-8 and their collaries require no proofs: they follow easily from Theorem 3 and the observations made in Section 1.

3. COMPARISONS WITH PREVIOUS RESULTS

The size of an instance of the maximum-weight clique problem is, roughly speaking, the amount of space required to record the data:

$$\text{size} = n + m + \sum_{i=1}^{n} \log_{10}(1 + w_i)$$

with n standing for the number of vertices, m for the number of edges, and w_i for the weight of the i-th vertex. An algorithm for solving the MWCP with input graphs restricted to some class C is referred to as a polynomial-time algorithm if, for some constant t, it solves any MWCP on any G in C in $O(\text{size}^t)$ steps. We shall now discuss classes C for which such algorithms have been designed; for each of these classes C, we shall point out graphs that belong to $TR^3 \cap TR^4$ but do not belong to C.

First, a graph is called perfect if, for each of its induced subgraphs F, the chromatic number of F equals the largest size of a clique in F. A polynomial-time algorithm for solving the MWCP on perfect graphs has been designed by Grötschel, Lovász, and Schrijver (1984a). Chordless cycles whose length is odd and at least five are not perfect, but the do belong to $TR^3 \cap TR^4$.

Second, Grötschel, Lovász and Schrijver (1984b) also designed a polynomial-time algorithm for solving the MWCP on complements of h-perfect graphs, defined as graphs for which the convex hull of the incidence vectors of stable sets is given by the clique inequalities, the odd cycle inequalities, and the nonnegativity conditions. The graph shown in Fig. 2 is not h-perfect, but its complement belongs to $TR^3 \cap TR^4$.

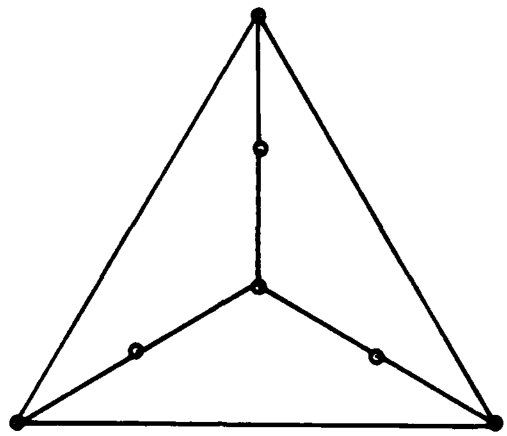

Figure 2

Third, a graph is called claw-free if it contains no induced subgraph with vertices x,y,z,w, whose edges are precisely xw,yw,zw. Minty (1980) designed a polynomial-time algorithm for solving the MWCP on complements of claw-free graphs (the unweighted case was settled independently by Sbihi (1978)). Graphs consisting of two vertex-disjoint cliques with at least five vertices altogether are not complements of claw-free graphs, but they trivially belong to $TR^3 \cap TR^4$.

Fourth, Hsu, Ikura and Nemhauser (1981) designed a polynomial-time algorithm for solving the MWCP on graphs whose complements contain no odd cycle longer than an arbitrary but fixed constant k. Graphs with more than k+1 vertices and with no edges at all do not belong to this class, but they trivially belong to $TR^3 \cap TR^4$.

Finally, Fact 1 implies that for every t-bounded class C there is an algorithm that solves any MWCP on any graph in C in $O(n^{2t+2})$ steps. To see that no t-bounded class contains $TR^3 \cap TR^4$, consider the sequence of graphs

$G_1, G_2, G_3, \ldots$ such that G_k has vertices $v_1, v_2, \ldots, v_{2k}$ and the only nonadjacent pairs are $v_1v_2, v_3v_4, \ldots, v_{2k-1}v_{2k}$. It is easy to see that $G_k \in TR^3 \cap TR^4$ for all k. On the other hand, G_k has 2^k maximal cliques, and so no t-bounded class contains all G_k.

4. EXTENSIONS, LIMITATIONS, APPLICATIONS

The observation underlying our method is that any MWCP on a graph G can be solved quickly whenever the vertex set of G is covered by a small number of sets $S_1, S_2, \ldots, S_N$ such that

(i) each clique in G is a subset of some S_i,

(ii) each S_i induces in G the complement of a bipartite graph.

(In Fact 3, we have $N = \binom{k}{2}$ and each S_r is some $Q_i \cup Q_j$.) The only reason for featuring complements of bipartite graphs in (ii) is that any MWCP on any of these graphs can be solved quickly; we could just as well use any other class C of graphs such that, for some constant t, some algorithm solves the MWCP on any G in C in $O(\text{size}^t)$ steps. We shall refer to such classes C as <u>t-solvable</u>; their examples include the four classes discussed in the preceding section (perfect graphs, complements of h-perfect graphs, complements of claw-free graphs, and complements of graphs with no odd cycle longer than a constant), as well as the classes $TR^3, \ldots, TR^6$ introduced in Section 2. To generalize Fact 3, let us define a <u>C-cover</u> in G as any collection of subsets $S_1, S_2, \ldots, S_k$ of the vertex set of G such that each clique in G is a subset of some S_i, and such that each S_i induces in G a member of C. The generalization goes as follows:

FACT 4. For every t-solvable class C there is an algorithm that, given any graph G along with a C-cover $S_1, S_2, \ldots, S_k$ solves any MWCP on G in $O(k\ \text{size}^t)$ steps.||

Now let $f(G,C)$ stand for the smallest k such that G admits a C-cover $S_1, S_2, \ldots, S_k$. Clearly, Fact 4 provides a polynomial-time algorithm for solving every MWCP on every graph in some class C^* only if, for some t-solvable class C and for all G in C^*,

$$f(G,C) \text{ does not exceed a fixed polynomial in } n. \tag{4.1}$$

We are going to show that, for every class C that satisfies a certain technical assumption, a randomly chosen graph G is extremely unlikely to satisfy (4.1). To make this claim precise, let us first clarify the meaning of "extremely unlikely". For this purpose, consider any property P that a graph may or may not have, and let $P(n)$ equal the number of graphs with vertices $v_1, v_2, \ldots, v_n$ that have the property.

It is customary to say that almost all graphs have property P if

$$\lim_{n \to \infty} \frac{P(n)}{2^{n(n-1)/2}} = 1 \tag{4.2}$$

(Observe that the denominator in (4.2) counts the number of graphs with vertices $v_1, v_2, \ldots, v_n$, and so the ratio in (4.2) equals the probability that a randomly chosen graph with these n vertices has property P.)

Throughout the remainder of this section, we write $\log x$ for $\log_2 x$.

THEOREM 9. Let C be any hereditary class of graphs other than the class of all graphs. Then for every positive ε, almost all graphs G have

$$f(G,C) > n^{(\frac{1}{2} - \varepsilon) \log n}. \quad ||$$

Our proof of Theorm 9 is based on two lemmas.

LEMMA 1. For every graph F there is a constant c such that almost all graphs G have the following property: every induced subgraph of G with at least $c \log n$ vertices contains an induced subgraph isomorphic to F.

PROOF. Let k stand for the number of vertices of F. Writing

$$a = \exp\left(\left(\tfrac{1}{2}\right)^{k(k-1)/2} \frac{1}{2k^2}\right),$$

we shall prove the statement with $c = 2/\log a$. To begin, let $p(s)$ denote the probability that a randomly chosen graph with vertices $w_1, w_2, \ldots, w_s$ has no induced subgraph isomorphic to F. We only need to show that

$$\lim_{n\to\infty} \binom{n}{\lceil c \log n\rceil} p(\lceil c \log n\rceil) = 0.$$

For this purpose, let $t(k,s)$ stand for the largest number t of sets $Q_1, Q_2, \ldots, Q_t$ such that

$$|Q_i| = k \text{ for all } i, \quad |Q_i \cap Q_j| \le 1 \text{ whenever } i \ne j \text{ and}$$

$$\left|\bigcup_{i=1}^{t} Q_i\right| \le s.$$

It is easy to see that

$$p(s) \le p(k)^{t(k,s)} \le \left(1 - \left(\tfrac{1}{2}\right)^{k(k-1)/2}\right)^{t(k,s)} \le \exp\left(-\left(\tfrac{1}{2}\right)^{k(k-1)/2} t(k,s)\right).$$

Erdős and Hanani (1963) have shown that

$$\lim_{s\to\infty} t(k,s). \frac{\binom{k}{2}}{\binom{s}{2}} = 1 \text{ for all } k;$$

In particular, $t(k,s) \ge s^2/2k(k-1) \ge s^2/2k^2$ whenever $s \ge s_0(k)$. Hence

$$p(s) \le a^{-s^2} \text{ whenever } s \ge s_0(k),$$

and so

$$\binom{n}{s} p(s) \le (na^{-s})^s \text{ whenever } s \ge s_0(k).$$

Since $a^{-c \log n} = n^{-2}$, the desired result follows.||

LEMMA 2. For every positive δ, almost all graphs G have at least

$$n^{(\frac{1}{2} - \delta)\log n}$$

cliques of size $\lfloor \log n \rfloor$.||

PROOF. As customary, we shall denote by P(A) the probability of event A, and we shall let E(X) stand for the expected value of a random variable X. We shall rely on the Chebyshev inequality, stating that

$$P(X \le E(X)-t) \le \frac{E(X^2) - (E(X))^2}{E(X^2) - (E(X))^2 + t^2} \tag{4.3}$$

In addition, we shall use the fact that

$$\sum_{i=0}^{k} \frac{\binom{k}{i}\binom{n-k}{k-i}}{\binom{n}{k}} t^i \le (1 + (t-1)\frac{k}{n})^k \text{ whenever } t \ge 1 \tag{4.4}$$

(for an elementary proof, see Chvátal (1979)).

Now let n and k be fixed and let a random variable X count the number of cliques of size k in a randomly chosen graph with vertices $v_1, v_2, \ldots, v_n$. Clearly,

$$E(X) = \binom{n}{k} (\tfrac{1}{2})^{\binom{k}{2}}$$

and

$$E(X^2) = \binom{n}{k} \sum_{i=0}^{k} \binom{k}{i}\binom{n-k}{k-i}\left(\frac{1}{2}\right)^{2\binom{k}{2} - \binom{i}{2}}$$

Note that

$$\frac{E(X^2)}{(E(X))^2} = \sum_{i=0}^{k} \frac{\binom{k}{i}\binom{n-k}{k-i}}{\binom{n}{k}} 2^{\binom{i}{2}}$$

and so (4.4) with $t = 2^{k/2}$ implies

$$1 \le \frac{E(X^2)}{(E(X))^2} \le \exp \frac{k^2 2^{k/2}}{n}.$$

Substituting into (4.3) we obtain

$$P(X \le \tfrac{1}{2} E(X)) \le 4(\exp \frac{k^2 2^{k/2}}{n} - 1). \tag{4.5}$$

In addition, note that

$$E(X) \ge \left(\frac{n-k}{k\ 2^{k/2}}\right)^k.$$

In particular, if $k = \lfloor \log n \rfloor$ then

$$\lim_{n \to \infty} \frac{\frac{1}{2} E(X)}{n^{(1/2-\delta)\log n}} = +\infty \text{ and } \lim_{n \to \infty} \exp \frac{k^2 2^{k/2}}{n} = 1,$$

and so the desired result follows from (4.5).||

PROOF OF THEOREM 9. By assumption, there is a graph F such that no graph in C contains an induced subgraph isomorphic to F. We only need show that all graphs G with the two properties specified in Lemma 1 and Lemma 2 have

$$f(G,C) > n^{(\frac{1}{2} - \delta)\log n - c}$$

For this purpose, consider an arbitrary C-cover $S_1, S_2, \ldots, S_k$ in G. W.l.o.g., we may assume that each S_i is minimal; then by Lemma 1, we have $|S_i| \le c \log n$ for all i. Now Lemma 2 implies

$$n^{(\frac{1}{2} - \delta)\log n} \le \sum_{i=1}^{k} 2^{|S_i|} \le k\, n^c,$$

which is the desired conclusion.||

Theorem 8 shows that for a randomly chosen graph G, Fact 4 is very unlikely to yield a polynomial-time algorithm for solving the MWCP on G. On the other hand, the results of this paper can be used to devise improved heuristics or enumerative (non-polynomial-time) algorithms for solving the MWCP on an arbitrary graph. Let C be any t-solvable class for some constant t, and suppose that for an arbitrary graph G, a maximal induced subgraph G(S) of G that belongs to C can be generated in polynomial time. This is the case, for instance, with the classes TR^1 and TR^2 introduced in Section 2. If Q is a maximum-weight clique of G(S), then any clique of larger weight than Q must contain some vertex of V\S. Denoting by N(v) the neighbor set of vertex v, one can branch by replacing G with the collection of induced subgraphs $G(N(v_1))$, $G(N(v_2)\backslash\{v_1\}), \ldots G(N(v_p)-\{v_1, \ldots, v_{p-1}\})$ where $\{v_1, \ldots, v_p\} = V\backslash S$. A branch and bound algorithm of this type, using as C the class of graphs whose chromatic number equals their maximum clique size, was proposed by Balas and Yu [1984] for the unweighted maximum clique problem. The algorithm was tested on randomly generated graphs with up to 400 vertices and 30,000 edges with considerably better results than earlier procedures based on straightforward

branch and bound. The classes of graphs introduced in this paper can be used in a similar fashion to obtain algorithms for the MWCP on general graphs.

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER: MSRR 518	2. GOVT ACCESSION NO. AD-A159953
3. RECIPIENT'S CATALOG NUMBER	
4. TITLE (and Subtitle): ON THE MAXIMUM-WEIGHT CLIQUE PROBLEM	5. TYPE OF REPORT & PERIOD COVERED: June 1985
	6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s): Egon Balas, Vašek Chvátal, Jaroslav Nešetřil	8. CONTRACT OR GRANT NUMBER(s): N00014-82-K-0329, NR047-607
9. PERFORMING ORGANIZATION NAME AND ADDRESS: Graduate School of Industrial Administration, Carnegie-Mellon University, Pittsburgh, Pennsylvania 15213	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS	12. REPORT DATE: June 1985
	13. NUMBER OF PAGES: 26
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)	15. SECURITY CLASS. (of this report)
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Maximum-Weight Clique
Vertex Packing
Triangulated Graphs
Triangle-Free Graphs
Polynomial-Time Algorithms

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

We introduce several new classes of graphs on which the maximum-weight clique problem is solvable in polynomial time. Their common feature, and the central idea of our algorithms, is that every clique of any of our graphs is contained in some member of a polynomial-sized collection of induced subgraphs that are complements of bipartite graphs. Our approach is quite general, and might conceivably yield many other classes of graphs along with corresponding polynomial time algorithms.

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